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On perfect packings in dense graphs

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Abstract

We say that a graph G has a perfect H -packing if there exists a set of vertex-disjoint copies of H which cover all the vertices in G . We consider various problems concerning perfect H -packings: Given $n, r, D \in \mathbb{N}$, we characterise the edge density threshold that ensures a perfect K_r -packing in any graph G on n vertices and with minimum degree $\delta(G) \geq D$. We also give two conjectures concerning degree sequence conditions which force a graph to contain a perfect H -packing. Other related embedding problems are also considered. Indeed, we give a structural result concerning K_r -free graphs that satisfy a certain degree sequence condition.

1 Introduction

Given two graphs H and G , a *perfect H -packing* in G is a collection of vertex-disjoint copies of H which cover all the vertices in G . Perfect H -packings are also referred to as *H -factors* or *perfect H -tilings*. Hell and Kirkpatrick [10] showed that the decision problem whether a graph G has a perfect H -packing is NP-complete precisely when H has a component consisting of at least 3 vertices. So for such graphs H , it is unlikely that there is a complete characterisation of those graphs containing a perfect H -packing. Thus, there has been significant attention on obtaining sufficient conditions that ensure a graph G contains a perfect H -packing.

A seminal result in the area is the Hajnal-Szemerédi theorem [9] which states that a graph G whose order n is divisible by r has a perfect K_r -packing provided that $\delta(G) \geq (r-1)n/r$. Kühn and Osthus [15, 16] characterised, up to an additive constant, the minimum degree which ensures a graph G contains a perfect H -packing for an arbitrary graph H .

It is easy to see that the minimum degree condition in the Hajnal-Szemerédi theorem cannot be lowered. Of course, this does not mean that one cannot strengthen this result. *Ore-type* degree conditions consider the sum of the degrees of non-adjacent vertices in a graph. The following Ore-type result of Kierstead and Kostochka [12] implies the Hajnal-Szemerédi theorem.

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Theorem 1 (Kierstead and Kostochka [12]) *Let $n, r \in \mathbb{N}$ such that r divides n . Suppose that G is a graph on n vertices such that for all non-adjacent $x \neq y \in V(G)$,*

$$d(x) + d(y) \geq 2(1 - 1/r)n - 1.$$

Then G contains a perfect K_r -packing.

Kühn, Osthus and Treglown [17] characterised, asymptotically, the Ore-type degree condition which ensures a graph G contains a perfect H -packing for an arbitrary graph H .

1.1 Perfect packings in dense graphs of low minimum degree

It is easy to characterise the edge density that forces a graph G to contain a perfect K_r -packing when there are no other restrictions. Indeed, given $n, r \in \mathbb{N}$ such that $r \geq 2$ divides n , if G is a graph on n vertices and $e(G) \geq \binom{n}{2} - n + r$ then G contains a perfect K_r -packing. Moreover, if G is a copy K of K_{n-1} together with a vertex which sends precisely $r - 2$ edges to K , then $e(G) = \binom{n}{2} - n + r - 1$ and G does not contain a perfect K_r -packing. The following result of Akiyama and Frankl [1] refines this observation.

Theorem 2 (Akiyama and Frankl [1]) *Let $n, r \in \mathbb{N}$ such that r divides n . Suppose G is a graph on n vertices and $e(\overline{G}) \leq \min\{\binom{n/r+1}{2}, n - r + 1\}$. Then G has a perfect K_r -packing unless \overline{G} is isomorphic to one of the following graphs:*

- (i) *A copy of $K_{n/r+1}$ together with $(1 - 1/r)n - 1$ isolated vertices;*
- (ii) *The disjoint union of $K_{1, n-r-j+1}$, j edges and $r-j-2$ isolated vertices, for some $1 \leq j \leq r-2$.*

When (for example) $n \geq r^3$, $\binom{n/r+1}{2} > n - r + 1$. Hence, in this case Theorem 2 is equivalent to the following: If G is a graph on n vertices and $e(G) \geq \binom{n}{2} - n + r - 1$ then either G contains a perfect K_r -packing or \overline{G} is isomorphic to a graph as in (ii).

In Sections 2 and 3 we consider the following natural problem: Let $n, r \in \mathbb{N}$ such that r divides n . Given some $D \in \mathbb{N}$, what edge density condition ensures that any graph G on n vertices and of minimum degree $\delta(G) \geq D$ contains a perfect K_r -packing?

We fully resolve the problem, and our answers for $r = 2$ and $r \geq 3$ differ.

Theorem 3 *For an even positive n and integer $1 \leq d < n/2$, let $h(n, d) := \binom{n-d-1}{2} + d(d+1)$ and let $f(2, n, d)$ denote the maximum integer c such that some n -vertex graph with minimum degree at least d and at least c edges has no perfect matching. Then*

$$f(2, n, d) = \max\{h(n, d), h(n, 0.5n - 1)\}.$$

Theorem 4 *Let $n, r \in \mathbb{N}$ such that $r \geq 3$ and r divides n . Given any $D \in \mathbb{N}$ such that $r - 1 \leq D \leq (r - 1)n/r - 1$ define*

$$g(n, r, D) := \max\left\{\binom{n}{2} - \binom{n/r+1}{2}, D(n - D) + \binom{n-1-D}{2} + e(T(D, r-2))\right\}.$$

Suppose that G is a graph on n vertices with $\delta(G) \geq D$ and $e(G) > g(n, r, D)$. Then G contains a perfect K_r -packing. Moreover, there exists a graph G' on n vertices with $\delta(G') \geq D$ and $e(G') = g(n, r, D)$ but such that G' does not contain a perfect K_r -packing.

Clearly a graph G of minimum degree $\delta(G) < r - 1$ cannot contain a perfect K_r -packing. Further, regardless of edge density, every graph G whose order n is divisible by r and with $\delta(G) \geq (r-1)n/r$ contains a perfect K_r -packing. Thus, Theorem 4 covers all values of D where our problem was not solved previously.

An *equitable k -colouring* of a graph G is a proper k -colouring of G such that any two colour classes differ in size by at most one. Let $n, r \in \mathbb{N}$ such that r divides n . Notice that a graph G on n vertices has a perfect K_r -packing if and only if the complement \overline{G} of G has an equitable n/r -colouring. So, for example, the Hajnal-Szemerédi theorem can be stated in terms of equitable colourings: Let G be a graph on n vertices such that r divides n . If $\Delta(G) \leq n/r - 1$ then G has an equitable n/r -colouring.

It is often easier to work in the language of equitable colourings compared to perfect packings. Indeed, rather than prove Theorem 1 directly, Kierstead and Kostochka proved the equivalent statement for equitable colourings. Here we also find it more convenient to work with equitable colourings. Thus, instead of proving Theorem 4 directly we prove the following equivalent result.

Theorem 5 *Let $n, r \in \mathbb{N}$ such that $r \geq 3$ and r divides n . Recall that $T(n, r)$ denotes the Turán graph. Given any $D \in \mathbb{N}$ such that $n/r \leq D \leq n - r$ define*

$$f(n, r, D) := \min \left\{ \binom{n/r + 1}{2}, D + e(\overline{T}(n - D - 1, r - 2)) \right\}.$$

Suppose that G is a graph on n vertices with $\Delta(G) \leq D$ and $e(G) < f(n, r, D)$. Then G has an equitable n/r -colouring. Moreover, there exists a graph G' on n vertices with $\Delta(G') \leq D$ and $e(G') = f(n, r, D)$ but such that G' does not have an equitable n/r -colouring.

We prove Theorem 3 and describe extremal constructions for Theorems 4 and 5 in Section 2. That is, we show that the edge density condition in Theorem 4 is best possible for all values of D such that $r - 1 \leq D \leq (r - 1)n/r - 1$. Section 3 contains a proof of Theorem 5.

1.2 Degree sequence conditions forcing a perfect packing

Chvátal [4] gave a condition on the degree sequence of a graph which ensures Hamiltonicity: Suppose that G is a graph on n vertices and that the degrees of the graph are $d_1 \leq \dots \leq d_n$. If $n \geq 3$ and $d_i \geq i + 1$ or $d_{n-i} \geq n - i$ for all $i < n/2$ then G is Hamiltonian. The following is a simple consequence of Chvátal's theorem.

Theorem 6 (Chvátal [4]) *Suppose that G is a graph on $n \geq 2$ vertices and the degrees of the graph are $d_1 \leq \dots \leq d_n$. If*

$$d_i \geq i \quad \text{or} \quad d_{n-i+1} \geq n - i \quad \text{for all} \quad 1 \leq i \leq n/2$$

then G contains a Hamilton path.

We propose the following conjecture on the degree sequence of a graph which forces a perfect K_r -packing.

Conjecture 7 *Let $n, r \in \mathbb{N}$ such that r divides n . Suppose that G is a graph on n vertices with degree sequence $d_1 \leq \dots \leq d_n$ such that:*

(α) $d_i \geq (r-2)n/r + i$ for all $i < n/r$;

(β) $d_{n/r+1} \geq (r-1)n/r$.

Then G contains a perfect K_r -packing.

Note that Conjecture 7, if true, is much stronger than the Hajnal-Szemerédi theorem since the degree condition allows for n/r vertices to have degree less than $(r-1)n/r$. Further, Proposition 17 in Section 4 shows that the condition on the degree sequence in Conjecture 7 is essentially “best possible”. It is easy to see that Theorem 6 implies Conjecture 7 in the case when $r = 2$. We prove the conjecture in the case when G is additionally K_{r+1} -free (see Section 5).

If one can prove Conjecture 7, it seems likely it can be used to prove the next conjecture.

Conjecture 8 *Suppose $\gamma > 0$ and H is a graph with $\chi(H) = r$. Then there exists an integer $n_0 = n_0(\gamma, H)$ such that the following holds. If G is a graph whose order $n \geq n_0$ is divisible by $|H|$, and whose degree sequence $d_1 \leq \dots \leq d_n$ satisfies*

- $d_i \geq (r-2)n/r + i + \gamma n$ for all $i < n/r$,

then G contains a perfect H -packing.

Since first submitting this paper, the third author and Knox [13] have proven Conjecture 8 in the case when $r = 2$. (In fact, they have proven a much more general result concerning embedding spanning bipartite graphs of small bandwidth.)

The following result of Erdős [8] characterises those degree sequences which force a copy of K_r in a graph G .

Theorem 9 (Erdős [8]) *Let G be a graph on n vertices with degree sequence $d_1 \leq \dots \leq d_n$. If G is K_{r+1} -free then there is an r -partite graph G' on n vertices whose degree sequence $d'_1 \leq \dots \leq d'_n$ satisfies*

$$d_i \leq d'_i \text{ for all } i \leq n.$$

In Section 6 we prove the following related structural theorem.

Theorem 10 *Suppose that $n, r \in \mathbb{N}$ such that $n \geq r$ and so that r divides n . Let G be a K_{r+1} -free graph on n vertices whose degree sequence $d_1 \leq \dots \leq d_n$ is such that $d_{n/r} \geq (r-1)n/r$. Then $G \subseteq T(n, r)$, where $T(n, r)$ is the complete r -partite Turán graph on n vertices; so each vertex class has size $\lceil n/r \rceil$ or $\lfloor n/r \rfloor$.*

2 The case $r = 2$ and extremal examples for $r \geq 3$

2.1 Perfect matchings in dense graphs

In this section we establish the density threshold that ensures every graph G on an even number n of vertices and of minimum degree $\delta(G) \geq d$ contains a perfect matching. Note that we only consider values of d such that $1 \leq d < n/2$, since if $\delta(G) \geq n/2$ then G has a perfect matching, regardless of the edge density.

Recall that $h(n, d) := \binom{n-d-1}{2} + d(d+1)$. Note that for a fixed even n , $h(n, d)$ decreases with d in the interval $[0, n/3 - 5/6]$ and increases with d in $[n/3 - 5/6, 0.5n - 1]$.

For a positive even n and an integer $0 \leq d < n/2$, let A, B and C be disjoint sets with $|A| = d+1$, $|B| = d$, $|C| = n - 2d - 1$. Let $H = H(n, d)$ be the graph with the vertex set $A \cup B \cup C$ such that $H[B \cup C] = K_{n-d-1}$, and each vertex in A is adjacent to each vertex in B and to no vertex in C . So H does not contain a perfect matching and has exactly $h(n, d)$ edges.

The examples of $H(n, d)$ show that $f(2, n, d) \geq \max\{h(n, d), h(n, 0.5n - 1)\}$. Thus to derive Theorem 3, it suffices to prove that an n -vertex graph G with $\delta(G) \geq d$ and $e(G) > \max\{h(n, d), h(n, 0.5n - 1)\}$ has a perfect matching.

Consider such a graph G . Let $d_1 \leq \dots \leq d_n$ denote the degree sequence of G . If $d_i \geq i$ for all $1 \leq i \leq n/2$ then Theorem 6 implies that G contains a perfect matching. Suppose for a contradiction that $d_{i'} \leq i' - 1$ for some $1 \leq i' \leq n/2$. Note that $i' > d$ as $\delta(G) \geq d$.

Let A denote the set of i' vertices in G that correspond to the first i' terms $d_1, \dots, d_{i'}$ of the degree sequence. Set $B := V(G) \setminus A$. Then

$$e(G[B]) \geq e(G) - i'(i' - 1) > \max\{h(n, d), h(n, 0.5n - 1)\} - i'(i' - 1)$$

since $d(x) \leq i' - 1$ for all $x \in A$. Note that $\max\{h(n, d), h(n, 0.5n - 1)\} \geq h(n, i' - 1)$ since $d < i' \leq n/2$. Therefore,

$$e(G[B]) > \max\{h(n, d), h(n, 0.5n - 1)\} - i'(i' - 1) \geq h(n, i' - 1) - i'(i' - 1) = \binom{n - i'}{2},$$

a contradiction as $|B| = n - i'$. Thus, $d_i \geq i$ for all $1 \leq i \leq n/2$, as desired.

2.2 Examples for $r \geq 3$

We will give the extremal examples for Theorem 5. Since Theorems 4 and 5 are equivalent, the complements of the extremal graphs for Theorem 5 are the extremal graphs for Theorem 4.

Proposition 11 *Suppose that $n, r \in \mathbb{N}$ such that $r \geq 3$ and r divides n . Then there exists a graph G_1 on n vertices such that $\Delta(G_1) = n/r$,*

$$e(G_1) = \binom{n/r + 1}{2},$$

but such that G_1 does not have an equitable n/r -colouring.

Proof. Let G_1 denote the disjoint union of a clique V on $n/r + 1$ vertices and an independent set W of $(1 - 1/r)n - 1$ vertices. So every independent set in G_1 contains at most one vertex from V . But since $|V| = n/r + 1$, G_1 does not have an equitable n/r -colouring. Further, $\Delta(G_1) = n/r$ and $e(G_1) = \binom{n/r + 1}{2}$. \square

Proposition 12 *Suppose that $n, r \in \mathbb{N}$ such that $r \geq 3$ and $n = kr$ for some $k \geq 2$. Further, let $D \in \mathbb{N}$ such that $n/(r - 1) \leq D \leq n - r$. Then there exists a graph G_2 on n vertices such that $\Delta(G_2) = D$,*

$$e(G_2) = D + e(\overline{T}(n - D - 1, r - 2)),$$

but such that G_2 does not have an equitable n/r -colouring.

Proof. Let G_2 denote the disjoint union of a copy K of $K_{1,D}$ and a copy of $\overline{T}(n - D - 1, r - 2)$. So $|G| = n$. Let v denote the vertex of degree D in K . The largest independent set in G_2 that contains v is of size $r - 1$. Thus, G_2 does not have an equitable n/r -colouring. Further, $e(G_2) = D + e(\overline{T}(n - D - 1, r - 2))$.

Since $n/(r - 1) \leq D$ we have that $n - 1 \leq (r - 1)D$. Thus, every vertex in the copy of $\overline{T}(n - D - 1, r - 2)$ has degree at most

$$\left\lceil \frac{n - D - 1}{r - 2} \right\rceil - 1 \leq \frac{n - D - 1}{r - 2} \leq D.$$

This implies that $\Delta(G_2) = D$. □

Clearly Propositions 11 and 12 show that one cannot lower the edge density condition in Theorem 5 in the case when $n/(r - 1) \leq D \leq n - r$. The following result, together with Proposition 11, shows that Theorem 5 is best possible in the case when $n/r \leq D \leq n/(r - 1)$.

Proposition 13 *Let $n, r \in \mathbb{N}$ such that $r \geq 3$ and r divides $n \geq 2r$. Suppose that $D \in \mathbb{N}$ such that $n/r \leq D \leq n/(r - 1)$. Then*

$$f(n, r, D) = \binom{n/r + 1}{2}.$$

The following simple consequence of Turán's theorem will be used in the proof of Theorem 5.

Fact 14 *Let $n, r \in \mathbb{N}$ such that $r \leq n$. Then*

$$e(T(n, r)) \leq \left(1 - \frac{1}{r}\right) \frac{n^2}{2} \quad \text{and thus} \quad e(\overline{T}(n, r)) \geq \frac{n^2}{2r} - \frac{n}{2}.$$

We will also require the following easy result.

Lemma 15 *Let $n, r \in \mathbb{N}$ such that $r \geq 4$ and r divides $n \geq 3r$. Suppose that $D \in \mathbb{N}$ such that $n/r \leq D < (n + r)/(r - 1)$. Then*

$$f(n, r, D) = \binom{n/r + 1}{2}.$$

3 Proof of Theorem 5

3.1 Preliminaries

Suppose for a contradiction that the result is false. Let G be a counterexample with the fewest vertices. That is, $n = |V(G)| = rk$ for some $k \in \mathbb{N}$, $\Delta(G) \leq D$ for some $D \in \mathbb{N}$ such that $n/r \leq D \leq n - r$, $e(G) < f(n, r, D)$ and G has no equitable n/r -colouring. By the Hajnal-Szemerédi theorem, $\Delta(G) \geq n/r$. Notice that given fixed n and r , $f(n, r, D)$ is non-increasing with respect to D . Thus, we may assume that $\Delta(G) = D$.

We first show that $k \geq 4$. Indeed, if $n = 2r$ then $f(n, r, D) \leq \binom{3}{2} = 3$. But it is easy to see that every graph G_1 on $2r$ vertices and with $e(G_1) \leq 2$ has an equitable 2-colouring. If $n = 3r$ then $f(n, r, D) \leq \binom{4}{2} = 6$. Consider any graph G_1 on $3r$ vertices with $e(G_1) \leq 5$ and $3 \leq \Delta(G_1) \leq 5$. Let x denote the vertex in G_1 where $d_{G_1}(x) = \Delta(G_1)$. Since $3 \leq d_{G_1}(x) \leq 5$, x lies in an independent

set I in G_1 of size r . But then $G_1 - I$ contains $2r$ vertices and at most 2 edges. So $G_1 - I$ has an equitable 2-colouring and hence G_1 has an equitable 3-colouring.

Let $v \in V(G)$ such that $d_G(v) = D$. Set $G^* := G - (N_G(v) \cup \{v\})$. Since $f(n, r, D) \leq D + e(\overline{T}(n - D - 1, r - 2))$ we have that $e(G^*) < e(\overline{T}(n - D - 1, r - 2))$. Thus, by Turán's theorem, G^* contains an independent set of size $r - 1$. Hence, v lies in an independent set in G of size r . Amongst all such independent sets of size r that contain v , choose a set $I = \{v, x_1, \dots, x_{r-1}\}$ such that $d_G(x_1) + \dots + d_G(x_{r-1})$ is maximised.

Set $G' := G - I$, $n' := |V(G')| = n - r$ and $D' := \Delta(G') \leq D$. Notice that $D' \geq n'/r$. (Indeed, if not, then by the Hajnal-Szemerédi theorem G' contains an equitable n'/r -colouring. Thus, as I is an independent set in G this gives us an equitable n/r -colouring of G , a contradiction.) Furthermore, $D' \leq n' - r$. If not then

$$e(G) \geq D + D' \geq 2D' \geq 2(n' - r + 1) = 2n - 4r + 2$$

and further,

$$\begin{aligned} e(G) &< f(n, r, D) \leq f(n, r, n - 2r + 1) \leq (n - 2r + 1) + e(\overline{T}(2r - 2, r - 2)) \\ &\leq (n - 2r + 1) + (r + 3) = n - r + 4. \end{aligned}$$

Therefore, $2n - 4r + 2 < n - r + 4$ and so $n < 3r + 2$ a contradiction since $n = kr \geq 4r$.

Since $n'/r \leq D' \leq n' - r$, if $e(G') < f(n', r, D')$ then the minimality of G implies that G' has an equitable n'/r -colouring. This then implies that G has an equitable n/r -colouring, a contradiction. Thus,

$$e(G') \geq f(n', r, D'). \quad (1)$$

We now split our argument into three cases.

3.2 Case 1: $f(n', r, D') = \binom{n'/r+1}{2}$.

By (1), $e(G') \geq \binom{n'/r+1}{2} = \binom{n/r}{2}$. Since $d_G(v) = D \geq n/r$,

$$e(G) \geq \frac{n}{r} + \binom{n/r}{2} = \binom{n/r+1}{2} \geq f(n, r, D),$$

a contradiction, as desired.

3.3 Case 2: $D' \leq D - 1$ and $f(n', r, D') = D' + e(\overline{T}(n' - D' - 1, r - 2))$.

The following claim will be useful.

Claim 16 $D' < \frac{r-1}{2r-3}n - \frac{(r^2-r+1)}{2r-3}$.

Proof. Note that

$$D + D' + e(\overline{T}(n' - D' - 1, r - 2)) \stackrel{(1)}{\leq} e(G) < f(n, r, D) \leq D + e(\overline{T}(n - D - 1, r - 2)). \quad (2)$$

Since $D' \leq D - 1$, clearly $e(\overline{T}(n' - D, r - 2)) \leq e(\overline{T}(n' - D' - 1, r - 2))$. Thus, (2) implies that

$$D' + e(\overline{T}(n' - D, r - 2)) < e(\overline{T}(n - D - 1, r - 2)). \quad (3)$$

One can obtain $\overline{T}(n - D - 1, r - 2)$ from $\overline{T}(n' - D, r - 2)$ by adding $r - 1$ vertices and at most

$$(n' - D) + \frac{n - D - 2}{r - 2} \text{ edges.} \quad (4)$$

Hence (3) and (4) give

$$D' < n' - D + \frac{n - D - 2}{r - 2}.$$

Rearranging, and using that $D' \leq D - 1$ and $n' = n - r$ we get that

$$\left(2 + \frac{1}{r - 2}\right) D' < \left(1 + \frac{1}{r - 2}\right) n - \frac{(r^2 - r + 1)}{r - 2}.$$

Thus,

$$D' < \frac{r - 1}{2r - 3} n - \frac{(r^2 - r + 1)}{2r - 3},$$

as desired. \square

Since we are in Case 2 we have that

$$D' + e(\overline{T}(n - r - D' - 1, r - 2)) \leq \binom{n'/r + 1}{2} = \binom{n/r}{2}. \quad (5)$$

Notice that for fixed n and r , $D' + e(\overline{T}(n - r - D' - 1, r - 2))$ is non-increasing as D' increases. Hence, (5) and Claim 16 imply that

$$D'' + e(\overline{T}(n - r - D'' - 1, r - 2)) \leq \frac{n^2}{2r^2} - \frac{n}{2r} \quad (6)$$

where $D'' := \lfloor (r - 1)n/(2r - 3) - (r^2 - r + 1)/(2r - 3) \rfloor$. Note that

$$n - r - \frac{r - 1}{2r - 3} n + \frac{(r^2 - r + 1)}{2r - 3} - 1 = \frac{r - 2}{2r - 3} n + \frac{4 - r^2}{2r - 3}.$$

So Fact 14 and (6) imply that

$$\begin{aligned} & \left(\frac{r - 1}{2r - 3} n - \frac{(r^2 - r + 1)}{2r - 3} - \frac{(2r - 4)}{2r - 3} \right) + \frac{1}{2(r - 2)} \left(\frac{r - 2}{2r - 3} n + \frac{4 - r^2}{2r - 3} \right)^2 \\ & - \frac{1}{2} \left(\frac{r - 2}{2r - 3} n + \frac{4 - r^2}{2r - 3} \right) \leq \frac{n^2}{2r^2} - \frac{n}{2r}. \end{aligned}$$

Next we will move all terms from the previous equation to the left hand side and simplify. The coefficient of n^2 is

$$\frac{r - 2}{2(2r - 3)^2} - \frac{1}{2r^2} = \frac{r^3 - 6r^2 + 12r - 9}{2r^2(2r - 3)^2}. \quad (7)$$

The coefficient of n is

$$\frac{r-1}{2r-3} - \frac{(r-2)}{2(2r-3)} + \frac{1}{2r} + \frac{(4-r^2)}{(2r-3)^2} = \frac{r^2-4r+9}{2r(2r-3)^2}. \quad (8)$$

The constant term is

$$-\frac{(r^2+r-3)}{2r-3} + \frac{(r^2-4)^2}{2(r-2)(2r-3)^2} + \frac{(r^2-4)}{2(2r-3)} = \frac{-r^4+3r^3+4r^2-26r+28}{2(r-2)(2r-3)^2}. \quad (9)$$

Since $n \geq 4r$, (7)–(9) imply that

$$\frac{8(r^3-6r^2+12r-9)}{(2r-3)^2} + \frac{2(r^2-4r+9)}{(2r-3)^2} + \frac{-r^4+3r^3+4r^2-26r+28}{2(r-2)(2r-3)^2} \leq 0. \quad (10)$$

Multiplying (10) by $2(r-2)(2r-3)^2$ we get

$$15r^4 - 121r^3 + 364r^2 - 486r + 244 \leq 0$$

This yields a contradiction, since it is easy to check that

$$15r^4 - 121r^3 + 364r^2 - 486r + 244 > 0$$

for all $r \in \mathbb{N}$ such that $r \geq 3$.

3.4 Case 3: $\mathbf{D}' = \mathbf{D}$ and $\mathbf{f}(\mathbf{n}', \mathbf{r}, \mathbf{D}') = \mathbf{D}' + \mathbf{e}(\overline{\mathbf{T}}(\mathbf{n}' - \mathbf{D}' - \mathbf{1}, \mathbf{r} - \mathbf{2}))$.

By (1) we have that

$$e(G') \geq f(n', r, D') = D' + e(\overline{\mathbf{T}}(n' - D' - 1, r - 2)). \quad (11)$$

Consider any vertex $x \in V(G')$ such that $d_{G'}(x) = D' = D$. Since $\Delta(G) = D$, x is not adjacent to any vertex in $I = \{v, x_1, \dots, x_{r-1}\}$. Further, I was chosen such that $d_G(x_1) + \dots + d_G(x_{r-1})$ is maximised. Thus, $d_G(x_1) = \dots = d_G(x_{r-1}) = D$. Together with (11) this implies that

$$e(G) \geq (r+1)D + e(\overline{\mathbf{T}}(n' - D - 1, r - 2)). \quad (12)$$

Since $e(G) < f(n, r, D) \leq D + e(\overline{\mathbf{T}}(n - D - 1, r - 2))$, (12) implies that

$$rD + e(\overline{\mathbf{T}}(n' - D - 1, r - 2)) < e(\overline{\mathbf{T}}(n - D - 1, r - 2)). \quad (13)$$

One can obtain $\overline{\mathbf{T}}(n - D - 1, r - 2)$ from $\overline{\mathbf{T}}(n' - D - 1, r - 2)$ by adding r vertices and at most

$$(n' - D - 1) + \frac{2(n - D - 3)}{r - 2} + 1 \text{ edges.} \quad (14)$$

Thus, (13) and (14) imply that

$$rD < n - r - D + \frac{2(n - D - 3)}{r - 2}$$

and so

$$\left(r + 1 + \frac{2}{r-2}\right) D < \left(1 + \frac{2}{r-2}\right) n + \frac{(-r^2 + 2r - 6)}{r-2} < \left(1 + \frac{2}{r-2}\right) n. \quad (15)$$

If $r = 3$ then (15) implies that

$$D < \frac{n}{2}.$$

Since $f(n', 3, D) = \min\{\binom{n'/3+1}{2}, D + \binom{n'-D-1}{2}\}$ it is easy to see that if $f(n', 3, D) = D + \binom{n'-D-1}{2}$ then $D \geq 2n'/3 + 1 = 2n/3 - 1$. Thus, $2n/3 - 1 \leq D < n/2$, a contradiction since $n \geq 4r = 12$.

If $r \geq 4$ then (15) implies that

$$D < \frac{n}{r-1} = \frac{n'}{r-1} + \frac{r}{r-1}.$$

Since $n' \geq 3r$, Lemma 15 implies that $f(n', r, D') = \binom{n'/r+1}{2}$ and so we are in Case 1, which we have already dealt with.

4 The extremal examples for Conjecture 7

Proposition 17 *Suppose that $n, r, k \in \mathbb{N}$ such that $r \geq 2$ divides n and $1 \leq k < n/r$. Then there exists a graph G on n vertices whose degree sequence $d_1 \leq \dots \leq d_n$ satisfies*

- $d_i = (r-2)n/r + k - 1$ for all $1 \leq i \leq k$;
- $d_i = (r-1)n/r$ for all $k+1 \leq i \leq (r-2)n/r + k$;
- $d_i = n - k - 1$ for all $(r-2)n/r + k + 1 \leq i \leq n - k + 1$;
- $d_i = n - 1$ for all $n - k + 2 \leq i \leq n$,

but such that G does not contain a perfect K_r -packing.

Proof. Let G' denote the complete $(r-2)$ -partite graph whose vertex classes V_1, \dots, V_{r-2} each have size n/r . Obtain G from G' by adding the following vertices and edges: Add a set V_{r-1} of $2n/r - 2k + 1$ vertices to G' , a set V_r of $k - 1$ vertices and a set V_0 of k vertices. Add all edges from $V_0 \cup V_{r-1} \cup V_r$ to $V_1 \cup \dots \cup V_{r-2}$. Further, add all edges with both endpoints in $V_{r-1} \cup V_r$. Add all possible edges between V_0 and V_r .

So V_0 is an independent set, and there are no edges between V_0 and V_{r-1} . This implies that any copy of K_r in G containing a vertex from V_0 must also contain at least one vertex from V_r . But since $|V_0| > |V_r|$ this implies that G does not contain a perfect K_r -packing. Furthermore, G has our desired degree sequence. \square

Notice that the graphs G considered in Proposition 17 satisfy (β) from Conjecture 7 and only fail to satisfy (α) in the case when $i = k$ (and in this case $d_k = (r-2)n/r + k - 1$).

Let $n, r \in \mathbb{N}$ such that r divides n . Denote by $T^*(n, r)$ the complete r -partite graph on n vertices with $r-2$ vertex classes of size n/r , one vertex class of size $n/r - 1$ and one vertex class of size $n/r + 1$. Then $T^*(n, r)$ does not contain a perfect K_r -packing. Furthermore, $T^*(n, r)$ satisfies (α) but condition (β) fails; we have that $d_{n/r+1} = (r-1)n/r - 1$ here. Thus, together $T^*(n, r)$ and Proposition 17 show that, if true, Conjecture 7 is essentially best possible.

5 A special case of Conjecture 7

We now give a simple proof of Conjecture 7 in the case when G is K_{r+1} -free.

Theorem 18 *Let $n, r \in \mathbb{N}$ such that $r \geq 2$ divides n . Suppose that G is a graph on n vertices with degree sequence $d_1 \leq \dots \leq d_n$ such that:*

- $d_i \geq (r-2)n/r + i$ for all $i < n/r$;
- $d_{n/r+1} \geq (r-1)n/r$.

Further suppose that no vertex $x \in V(G)$ of degree less than $(r-1)n/r$ lies in a copy of K_{r+1} . Then G contains a perfect K_r -packing.

Proof. We prove the theorem by induction on n . In the case when $n = r$ then $d_{n/r+1} = d_2 \geq (r-1)r/r = r-1$. This implies that every vertex in G has degree $r-1$. Hence $G = K_r$ as desired. So suppose that $n > r$ and the result holds for smaller values of n . Let $x_1 \in V(G)$ such that $d_G(x_1) = d_1 \geq (r-2)n/r + 1$. If $d_G(x_1) \geq (r-1)n/r$ then $\delta(G) \geq (r-1)n/r$. Thus G contains a perfect K_r -packing by the Hajnal-Szemerédi theorem. So we may assume that $(r-2)n/r + 1 \leq d_G(x_1) < (r-1)n/r$. In particular, x_1 does not lie in a copy of K_{r+1} . We first find a copy of K_r containing x_1 . If $r = 2$, x_1 has a neighbour and so we have our desired copy of K_2 . So assume that $r \geq 3$. Certainly $N_G(x_1)$ contains a vertex x_2 such that $d_G(x_2) \geq (r-1)n/r$. Thus, $|N_G(x_1) \cap N_G(x_2)| \geq (r-3)n/r + 1 > 0$. So if $r = 3$ we obtain our desired copy of K_r . Otherwise, we can find a vertex $x_3 \in N_G(x_1) \cap N_G(x_2)$ such that $d_G(x_3) \geq (r-1)n/r$. We can repeat this argument until we have obtained vertices x_1, \dots, x_r that together form a copy K'_r of K_r .

Let $G' := G - V(K'_r)$ and set $n' := n - r = |V(G')|$. Since G does not contain a copy of K_{r+1} containing x_1 , every vertex $x \in V(G) \setminus V(K'_r)$ sends at most $r-1$ edges to K'_r in G . Thus, $d_{G'}(x) \geq d_G(x) - (r-1)$ for all $x \in V(G')$. So if $d_G(x) \geq (r-1)n/r$ then $d_{G'}(x) \geq (r-1)n/r - (r-1) = (r-1)n'/r$ for all $x \in V(G')$. If a vertex $y \in V(G')$ does not lie in a copy of K_{r+1} in G then clearly y does not lie in a copy of K_{r+1} in G' . This means that no vertex $y \in V(G')$ of degree less than $(r-1)n'/r$ lies in a copy of K_{r+1} .

Let $d'_1 \leq \dots \leq d'_{n'}$ denote the degree sequence of G' . It is easy to check that $d'_i \geq (r-2)n'/r + i$ for all $i < n'/r$ and that $d'_{n'/r+1} \geq (r-1)n'/r$. Indeed, since $x_1 \in V(K'_r)$ where $d_G(x_1) = d_1$, we have that $d'_i \geq d_{i+1} - (r-1)$ for all $1 \leq i \leq n'$. Thus, for all $1 \leq i < n'/r = n/r - 1$, $d'_i \geq d_{i+1} - (r-1) \geq (r-2)n/r + (i+1) - (r-1) = (r-2)n'/r + i$. Similarly, $d'_{n'/r+1} = d'_{n/r} \geq d_{n/r+1} - (r-1) \geq (r-1)n/r - (r-1) = (r-1)n'/r$. Hence, by induction G' contains a perfect K_r -packing. Together with K'_r this gives us our desired perfect K_r -packing in G . \square

6 Proof of Theorem 10

Consider any $x_1 \in V(G)$ such that $d_G(x_1) \geq (r-1)n/r$. Since $d_{n/r} \geq (r-1)n/r$ we can greedily select vertices x_2, \dots, x_{r-1} such that

- x_1, \dots, x_{r-1} induce a copy of K_{r-1} in G ;
- $d_G(x_i) \geq (r-1)n/r$ for all $1 \leq i \leq r-1$.

Note that since G is K_{r+1} -free, $\cap_{i=1}^{r-1} N_G(x_i)$ is an independent set. The choice of x_1, \dots, x_{r-1} implies that $|\cap_{i=1}^{r-1} N_G(x_i)| \geq n/r$. Let V_1 denote a subset of $\cap_{i=1}^{r-1} N_G(x_i)$ of size n/r . Thus V_1 contains a vertex x_1^1 of degree at least $(r-1)n/r$.

As before we can find vertices x_2^1, \dots, x_{r-1}^1 such that

- x_1^1, \dots, x_{r-1}^1 induce a copy of K_{r-1} in G ;
- $d_G(x_i^1) \geq (r-1)n/r$ for all $1 \leq i \leq r-1$.

So $\cap_{i=1}^{r-1} N_G(x_i^1)$ is an independent set of size at least n/r . Let V_2 denote a subset of $\cap_{i=1}^{r-1} N_G(x_i^1)$ of size n/r . Note that $N_G(x_1^1) \cap V_1 = \emptyset$ since $x_1^1 \in V_1$ and V_1 is an independent set. Thus as $V_2 \subseteq N_G(x_1^1)$, $V_1 \cap V_2 = \emptyset$.

Our aim is to find disjoint sets $V_1, \dots, V_r \subseteq V(G)$ of size n/r and vertices $x_1^1, \dots, x_{r-1}^1, x_1^2, \dots, x_{r-1}^2, \dots, x_1^{r-1}, \dots, x_{r-1}^{r-1}$ with the following properties:

- $G[V_j]$ is an independent set for all $1 \leq j \leq r$;
- Given any $1 \leq j \leq r-1$, $x_k^j \in V_k$ for each $1 \leq k \leq j$;
- $d_G(x_k^j) \geq (r-1)n/r$ for all $1 \leq j \leq r-1$ and $1 \leq k \leq r-1$;
- x_1^j, \dots, x_{r-1}^j induce a copy of K_{r-1} in G for all $1 \leq j \leq r-1$.

Clearly finding such a partition V_1, \dots, V_r of $V(G)$ implies that $G \subseteq T(n, r)$.

Suppose that for some $1 < j < r$ we have defined sets V_1, \dots, V_j and vertices $x_1^1, \dots, x_{r-1}^1, \dots, x_1^{j-1}, \dots, x_{r-1}^{j-1}$ with our desired properties. Since $d_{n/r} \geq (r-1)n/r$ and V_1, \dots, V_j are independent sets of size n/r we can choose vertices x_1^j, \dots, x_{r-1}^j such that for all $1 \leq k \leq j$,

- $x_k^j \in V_k$ and $d_G(x_k^j) \geq (r-1)n/r$.

This degree condition, together with the fact that x_1^j, \dots, x_{r-1}^j lie in different vertex classes, implies that these vertices form a copy of K_j in G . We now greedily select further vertices $x_{j+1}^j, \dots, x_{r-1}^j$ such that

- x_1^j, \dots, x_{r-1}^j induce a copy of K_{r-1} in G ;
- $d_G(x_k^j) \geq (r-1)n/r$ for all $j+1 \leq k \leq r-1$.

So $\cap_{i=1}^{r-1} N_G(x_i^j)$ is an independent set of size at least n/r . Let V_{j+1} denote a subset of $\cap_{i=1}^{r-1} N_G(x_i^j)$ of size n/r . Note that, for each $1 \leq k \leq j$, $N_G(x_k^j) \cap V_k = \emptyset$ since $x_k^j \in V_k$ and V_k is an independent set. Thus as $V_{j+1} \subseteq N_G(x_k^j)$ for each $1 \leq k \leq j$, V_{j+1} is disjoint from $V_1 \cup \dots \cup V_j$.

Repeating this argument we obtain our desired sets $V_1, \dots, V_r \subseteq V(G)$ and vertices $x_1^1, \dots, x_{r-1}^1, x_1^2, \dots, x_{r-1}^2, \dots, x_1^{r-1}, \dots, x_{r-1}^{r-1}$.

7 Possible extensions of Conjecture 7

We suspect that the following ‘Chvátal-type’ degree sequence condition forces a graph to contain a perfect K_r -packing.

Question 19 Let $n, r \in \mathbb{N}$ such that $r \geq 2$ divides n . Suppose that G is a graph on n vertices with degree sequence $d_1 \leq \dots \leq d_n$ such that for all $i \leq n/r$:

- $d_i \geq (r-2)n/r + i$ or $d_{n-i(r-1)+1} \geq n-i$.

Does this condition imply that G contains a perfect K_r -packing?

Note that Theorem 6 answers this question in the affirmative when $r = 2$. The following example shows that we cannot have a lower value in the second part of the condition in Question 19.

Proposition 20 Suppose that $n, r, k \in \mathbb{N}$ such that $r \geq 2$ divides n and $1 \leq k \leq n/r$. Then there exists a graph G on n vertices whose degree sequence $d_1 \leq \dots \leq d_n$ satisfies

- $d_{n-i(r-1)+1} \geq n-i$ for all $i \in [n/r] \setminus \{k\}$;
- $d_{n-k(r-1)+1} = n-k-1$,

but such that G does not contain a perfect K_r -packing.

Proof. Let G be the graph on n vertices with vertex classes V_1, V_2 and V_3 of sizes $k, (r-1)k-1$ and $n-rk+1$ respectively and with the following edges: There are all possible edges between V_1 and V_2 and between V_2 and V_3 . Further add all possible edges in V_2 and all edges in V_3 . Thus, V_1 is an independent set and there are no edges between V_1 and V_3 .

The degree sequence of G is

$$\underbrace{(r-1)k-1, \dots, (r-1)k-1}_k, \underbrace{n-k-1, \dots, n-k-1}_{n-rk+1 \text{ times}}, \underbrace{n-1, \dots, n-1}_{(r-1)k-1 \text{ times}}.$$

Hence G satisfies our desired degree sequence condition. Every copy K'_r of K_r in G that contains a vertex from V_1 must contain $r-1$ vertices from V_2 . But since $|V_1|(r-1) > |V_2|$ this implies that G does not contain a perfect K_r -packing. \square

The r th power of a Hamilton cycle C is obtained from C by adding an edge between every pair of vertices of distance at most r on C . Seymour [18] conjectured the following strengthening of Dirac's theorem.

Conjecture 21 (Pósa-Seymour, see [18]) Let G be a graph on n vertices. If $\delta(G) \geq \frac{r}{r+1}n$ then G contains the r th power of a Hamilton cycle.

Pósa (see [7]) had earlier proposed the conjecture in the case of the square of a Hamilton cycle (that is, when $r = 2$). Komlós, Sárközy and Szemerédi [14] proved Conjecture 21 for graphs whose order is sufficiently larger than r . More recently, Cháu, DeBiasio and Kierstead [3] proved Pósa's conjecture for graphs of order at least 2×10^8 .

In the case when $r+1$ divides $|G|$, a necessary condition for a graph G to contain the r th power of a Hamilton cycle is that G contains a perfect K_{r+1} -packing. Further, notice that the minimum degree condition in Conjecture 21 is the same as the condition in the Hajnal-Szemerédi theorem with respect to perfect K_{r+1} -packings. Thus an obvious question is whether the condition in Conjecture 7 forces a graph to contain the $(r-1)$ th power of a Hamilton cycle. Interestingly though, when $r = 3$, this is not the case.

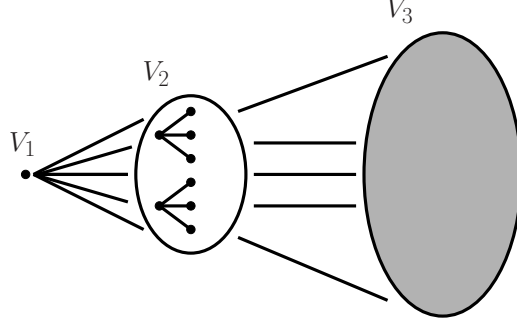


Figure 1: The example from Proposition 22 in the case when $K = 2$ and $|V_2| = 8$.

Proposition 22 *Suppose that $C, n \in \mathbb{N}$ such that $C \ll n$ and 3 divides n . Then there exists a graph G whose degree sequence $d_1 \leq \dots \leq d_n$ satisfies*

$$d_i \geq \frac{n}{3} + C + i \quad \text{for all } 1 \leq i \leq \frac{n}{3}$$

but such that G does not contain the square of a Hamilton cycle.

Proof. Choose $C, K, n \in \mathbb{N}$ so that $C \ll K \ll n$. Let G denote the graph on n vertices consisting of three vertex classes $V_1 = \{v\}$, V_2 and V_3 where $|V_2| = n/3 + C + 1$ and $|V_3| = 2n/3 - C - 2$ which contains the following edges:

- All edges from v to V_2 ;
- All edges between V_2 and V_3 and all possible edges in V_3 ;
- There are K vertex-disjoint stars in V_2 , each of size $\lfloor |V_2|/K \rfloor, \lceil |V_2|/K \rceil$, which cover all of V_2 (see Figure 1).

Let $d_1 \leq \dots \leq d_n$ denote the degree sequence of G . There are $n/3 + C - K + 1 \leq n/3 - 2C - 1$ vertices in V_2 of degree $2n/3 - C$. Since $C \ll K \ll n$, the remaining K vertices in V_2 have degree at least $2n/3 - C - 2 + \lfloor |V_2|/K \rfloor \geq 2n/3 + C + 1$. Since $d_G(v) = n/3 + C + 1$ and $d_G(x) = n - 2$ for all $x \in V_3$, we have that $d_i \geq \frac{n}{3} + C + i$ for all $1 \leq i \leq \frac{n}{3}$.

A necessary condition for a graph G to contain the square of a Hamilton cycle is that, for every $x \in V(G)$, $G[N(x)]$ contains a path of length 3. Note that $N(v) = V_2$ and $G[V_2]$ does not contain a path of length 3. So G does not contain the square of a Hamilton cycle. \square

Notice that we can set $C = o(\sqrt{n})$ in Proposition 22. We finish by raising the following question.

Question 23 *What can be said about degree sequence conditions which force a graph to contain the r th power of a Hamilton cycle? In particular, can one establish a degree sequence condition that ensures a graph G on n vertices contains the r th power of a Hamilton cycle and which allows for “many” vertices of G to have degree “much less” than $rn/(r+1)$?*

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Appendix

Here we give proofs of Proposition 13 and Lemma 15. The following fact will be used in both of these proofs.

Fact 24 Fix $n, r \in \mathbb{N}$ such that $r \geq 3$ and r divides $n \geq 2r$. Define

$$h(x) := x + \frac{(n-x-1)^2}{2(r-2)} - \frac{1}{2}(n-x-1).$$

Then $h(x)$ is a decreasing function for $x \in [0, n/(r-1)]$. Moreover, if $n \geq 3r$ then $h(x)$ is a decreasing function for $x \in [0, (n+r)/(r-1)]$.

Proof. Notice that

$$h'(x) = \frac{3}{2} - \frac{(n-x-1)}{r-2} = \frac{x}{r-2} + \frac{1-n}{r-2} + \frac{3}{2}.$$

So for $x \leq n/(r-1)$,

$$h'(x) \leq \frac{n}{(r-1)(r-2)} + \frac{1-n}{r-2} + \frac{3}{2} = -\frac{n}{r-1} + \frac{1}{r-2} + \frac{3}{2}.$$

Note that $3(r-1)/2 + (r-1)/(r-2) < n$ since $n \geq 2r$ and $r \geq 3$. Thus,

$$h'(x) \leq -\frac{n}{r-1} + \frac{1}{r-2} + \frac{3}{2} < 0.$$

If $x \leq (n+r)/(r-1)$ then

$$h'(x) \leq \frac{n+r}{(r-1)(r-2)} + \frac{1-n}{r-2} + \frac{3}{2} = -\frac{n}{r-1} + \frac{1}{r-2} + \frac{r}{(r-1)(r-2)} + \frac{3}{2}.$$

If $n \geq 3r$ then $n > 3r/2 + 4$. So $n > 3(r-1)/2 + (2r-1)/(r-2)$. Thus,

$$h'(x) \leq -\frac{n}{r-1} + \frac{1}{r-2} + \frac{r}{(r-1)(r-2)} + \frac{3}{2} < 0,$$

as desired. □

Proof of Proposition 13. We need to show that, for all $D \in \mathbb{N}$ such that $n/r \leq D \leq n/(r-1)$,

$$\frac{n^2}{2r^2} + \frac{n}{2r} = \binom{n/r + 1}{2} \leq D + e(\overline{T}(n - D - 1, r - 2)).$$

Since $D \leq n/(r-1)$, Facts 14 and 24 imply that

$$\begin{aligned} D + e(\overline{T}(n - D - 1, r - 2)) &\geq D + \frac{(n - D - 1)^2}{2(r - 2)} - \frac{(n - D - 1)}{2} \\ &\geq \frac{n}{r - 1} + \frac{1}{2(r - 2)} \left[\frac{(r - 2)}{r - 1} n - 1 \right]^2 - \frac{1}{2} \left[\frac{(r - 2)}{r - 1} n - 1 \right] \\ &\geq \frac{(r - 2)}{2(r - 1)^2} n^2 - \frac{(r - 2)}{2(r - 1)} n. \end{aligned}$$

Thus, it suffices to show that

$$\frac{(r - 2)}{2(r - 1)^2} n - \frac{r - 2}{2(r - 1)} \geq \frac{n}{2r^2} + \frac{1}{2r}. \quad (16)$$

Notice that

$$\frac{r - 2}{2(r - 1)^2} - \frac{1}{2r^2} = \frac{(r - 2)r^2 - (r - 1)^2}{2r^2(r - 1)^2} = \frac{r^3 - 3r^2 + 2r - 1}{2r^2(r - 1)^2} \quad (17)$$

and

$$\frac{r - 2}{2(r - 1)} + \frac{1}{2r} = \frac{r^2 - r - 1}{2r(r - 1)}.$$

Since $n \geq 2r$, (16) implies that it suffices to show that

$$\frac{r^3 - 3r^2 + 2r - 1}{r(r - 1)^2} - \frac{r^2 - r - 1}{2r(r - 1)} \geq 0. \quad (18)$$

Note that $r^3 \geq 4r^2 - 4r + 3$ as $r \geq 3$. Thus, $2(r^3 - 3r^2 + 2r - 1) \geq (r^2 - r - 1)(r - 1)$. So indeed (18) is satisfied, as desired. \square

Proof of Lemma 15. We need to show that, for all $D \in \mathbb{N}$ such that $n/r \leq D < (n + r)/(r - 1)$,

$$\frac{n^2}{2r^2} + \frac{n}{2r} = \binom{n/r + 1}{2} \leq D + e(\overline{T}(n - D - 1, r - 2)).$$

Since $D < (n + r)/(r - 1)$ we have that $D \leq n/(r - 1) + 1$. So Facts 14 and 24 imply that

$$\begin{aligned} D + e(\overline{T}(n - D - 1, r - 2)) &\geq D + \frac{(n - D - 1)^2}{2(r - 2)} - \frac{(n - D - 1)}{2} \\ &\geq \frac{n}{r - 1} + 1 + \frac{1}{2(r - 2)} \left[\frac{(r - 2)}{r - 1} n - 2 \right]^2 - \frac{1}{2} \left[\frac{(r - 2)}{r - 1} n - 2 \right] \\ &\geq \frac{(r - 2)}{2(r - 1)^2} n^2 - \frac{(r - 2)}{2(r - 1)} n - \frac{n}{r - 1}. \end{aligned}$$

Thus, it suffices to show that

$$\frac{(r-2)}{2(r-1)^2}n - \frac{(r-2)}{2(r-1)} - \frac{1}{r-1} \geq \frac{n}{2r^2} + \frac{1}{2r}. \quad (19)$$

Notice that

$$\frac{r-2}{2(r-1)} + \frac{1}{r-1} + \frac{1}{2r} = \frac{r^2 + r - 1}{2r(r-1)}.$$

Since $n \geq 3r$, (17) and (19) imply that it suffices to show that

$$\frac{3(r^3 - 3r^2 + 2r - 1)}{2r(r-1)^2} - \frac{r^2 + r - 1}{2r(r-1)} \geq 0. \quad (20)$$

Note that $2r^3 - 9r^2 + 8r - 4 \geq 0$ as $r \geq 4$. Thus, $3(r^3 - 3r^2 + 2r - 1) \geq (r^2 + r - 1)(r - 1)$. So indeed (20) is satisfied, as desired. \square